International Journal of Theoretical Physics, Vol. 30, No. 3, 1991

Invariance Transformations and Exact Solutions of the Rabi Model

K. A. Rustamov[1,2]

Received August 25, 1990

The invariance transformations of the Rabi model describing the interaction of a two-level system with an l-mode electromagnetic field are constructed. On this basis explicit expressions for the coherent states and Green function of the problem are obtained.

The interaction of an electromagnetic field with multilevel systems has been a subject of great interest in the last two decades. Many fascinating experimental as well as theoretical results have been presented in a number of original and survey publications (Louisell, 1974, 1977; Shimoda, 1986; de Olivera and Knight, 1988; Shumovsky *et al.*, 1986; Aliskendrov *et al.*, 1987; Rustamov, 1987; Yoo and Eberly, 1985; Bogolubov *et al.*, 1989).

However, most of the theoretical studies of analytical models are based on certain approximations, among them the widely accepted so-called rotating-wave approximation (see above references).

Nevertheless, according to experience in studies of the algebraic properties of multidimensional analytical models of condensed matter theory (and among them those concerned with the particular problems of electromagnetic wave interaction with a multilevel systems) (Rustamov, 1987, 1989*a,b*) one might suspect that the basic theoretical models of this subject, constructed from first principles, also seem to possess rather nontrivial properties of invariance which are the general specifications of the possibility of obtaining exact results.

So let us consider the Rabi model assumed to describe the interaction of an l-mode electromagnetic wave with two-level systems and whose

[1]International Centre for Theoretical Physics, Trieste, Italy.
[2]Permanent address: Azerbaijan SSR Academy of Sciences, Baku, USSR.

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Hamiltonian is ($\hbar = 1$)

$$H(a_k^+, a_k) = \omega_0\sigma_3 + \sum_{k=1}^{l} [w_k' a_k^+ a_k + (\sigma_+ + \sigma_-) g_k' (a_k^+ + a_k)] \tag{1}$$

where $k = 1, \ldots, l$; w_k' and g_k' are the frequencies of the electromagnetic wave and the interaction constants of each mode with the given system, respectively; we have

$$\sigma_\pm = \sigma_1 \pm i\sigma_2$$

$\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices; a_k^+ and a_k are the creation and annihilation operators for the kth mode of electromagnetic wave; and ω_0 is the specific frequency of the two-level system.

Let us call operators $Q_m(t, a_k^+, a_k)$ $(m = 1, \ldots; k = 1, \ldots, l;$ t is the time parameter) invariance operators of the dynamical problem under consideration (which means that these operators map the hypothetical manifold of solutions of the Schrödinger equation with a given Hamiltonian operator onto itself) if and only if the following condition is satisfied for all operators $Q_m(t, a_k^+, a_k)$ $(m = 1, \ldots)$:

$$\begin{aligned}&[i\,\partial/\partial t - H(a_k^+, a_k), Q_m(t, a_k^+, a_k)]\\ &\quad = \sigma(t, a_k^+, a_k)[i\,\partial/\partial t - H(a_k^+, a_k)]\end{aligned} \tag{2}$$

where $\sigma(t, a_k^+, a_k)$ is an arbitrary linear operator depending on the t, a_k^+, a_k and $[\xi, \mu] = \xi\mu - \mu\xi$.

Then it is rather easy to see that there are some invariance operators of the Rabi model, some of which might be represented by the following formula:

$$\begin{aligned}Q_k = \frac{1}{(2\rho_k)^{1/2}} \Big[&(n_{1,k}\, e^{2iw_k t} + n_{2,k}\, e^{-2iw_k t})(a_k^+ + a_k)\\ &+ (n_{1,k}\, e^{2iw_k t} - n_{2,k}\, e^{-2iw_k t})(a_k^+ - a_k)\\ &+ \frac{g_k}{w_k}(n_{1,k}\, e^{2iw_k t} + n_{2,k}\, e^{-2iw_k t})\Big] + \text{const}\end{aligned} \tag{3}$$

where

$$w_k = \frac{w_{k'}'}{2}; \qquad g_k = 2\sqrt{2}\,\sigma_1 g_k'$$

$$\rho_k = 2(|n_{1,k}|^2 - |n_{2,k}|^2)$$

$n_{1,k}$ and $n_{2,k}$ are arbitrary complex parameters. All of these operators Q_k $(k = 1, \ldots, l)$ and their Hermitian conjugates Q_k^+ satisfy the condition (2).

Furthermore, it is easy to see that dependences on the indices k of the parameters $n_{1,k}$ and $n_{2,k}$ and so of the ρ_k might be neglected.

Hence one may consider the following invariance operations of the Rabi problem:

$$Q_k = \frac{1}{(2\rho)^{1/2}}\Big[(n_1\, e^{2iw_k t} + n_2\, e^{-2iw_k t})(a_k^+ + a_k) + (n_1\, e^{2iw_k t} - n_2\, e^{-2iw_k t})(a_k^+ - a_k) + \frac{g_k}{w_k}(n_1\, e^{2iw_k t} + n_2\, e^{-2iw_k t})\Big] + \text{const} \tag{4}$$

It is important to mention that the operators (4) form the Heisenberg-Weyl algebra, namely

$$[Q_k, Q_{k'}^+] = I\delta_{kk'} \qquad (k, k' = 1, \ldots, l)$$
$$[Q_k, Q_{k'}] = [Q_k^+, Q_{k'}^+] = 0 \tag{5}$$

where δ_{kk} is the Kronecker symbol, and I is the operator of the identity transformation.

According to the general theory of the Glauber coherent states, having obtained the Heisenberg-Weyl algebra of invariance of the problem under consideration, it is easy to get expressions for these functions $|\nu\rangle$ (Malkin and Man'ko, 1979; Perelomov, 1983). This might be done by constructing the Glauber shift operator $D(\nu)$ and acting on the function $|\nu\rangle$, which satisfies the following condition (Rustamov, 1987; Malkin and Man'ko 1979; Perelomov, 1983):

$$Q_k|0\rangle = 0 \qquad (k = 1, \ldots, l) \tag{6}$$

Namely,

$$D(\nu) = \prod_{k=1}^{l} \exp(\nu_k Q_k^+ - \nu_k^* Q_k) = \prod_{k=1}^{l} \exp\left(\frac{-|\nu_k|^2}{2}\right) \exp(\nu_k Q_k^+) \exp(-\nu_k^* Q_k) \tag{7}$$

and

$$|\nu\rangle = D(\nu)|0\rangle$$

where the complex parameters ν_k $(k = 1, \ldots, l)$ are the eigenvalues of the operators Q_k $(k=1,\ldots,l)$, so that

$$Q_k|\nu\rangle = \nu_k|\nu\rangle$$

Thus, for the normalized coherent states of the Rabi model we obtain the following formula:

$$|\nu\rangle = L\exp\left\{\frac{\sqrt{2}}{2b}\sum_{k=1}^{l}\left[(\nu_k-d)(a_k^+ + a_k) - \frac{\lambda\sqrt{2}}{4}(a_k^+ + a_k)^2\right]\right\} \tag{8}$$

where

$$\lambda = \frac{1}{\sqrt{\rho}}(n_1\, e^{2iw_k t} + n_2\, e^{-2iw_k t})$$

$$b = \frac{1}{\sqrt{\rho}}(n_1\, e^{2iw_k t} - n_2\, e^{-2iw_k t})$$

$$d = \frac{g_k}{2w_k}\lambda + \text{const} \tag{9}$$

$$L = \pi^{-l/4}\left(\frac{b\lambda^* + \lambda b^*}{2bb^*}\right)^{l/4}$$

$$\times\exp\left(\sum_{k=1}^{l}\frac{-[b(\nu_k^* - d^*) + b^*(\nu_k - d)]^2}{4bb^*(\lambda b^* + \lambda^* b)}\right)$$

It is also an easy task to see that the functions $|\nu\rangle$ of (8) form an orthogonal and overcomplete set of functions, which is a general property of the Glauber coherent states (Malkin and Man'ko, 1979; Perelomov, 1983).

Then, according to the general formula for the Green function of the problem under consideration (Malkin and Man'ko, 1979; Perelomov, 1983),

$$G(a', t'; a, t) = \pi^{-l}\int_{-\infty}^{\infty}\langle a', t'|\nu\rangle\langle\nu|a, t\rangle\, d^2\nu \tag{10}$$

[where $d^2\nu = d(\mathrm{Re}\,\nu)\; d(\mathrm{Im}\,\nu)$], we obtain

$$G(a', t'; a, t)$$

$$= \pi^{-l}\prod_{k=1}^{l} C_1^{-1}\prod_{k=2}^{l}(C_7 - C_5^2)^{-1/2}$$

$$\times\exp\left\{\sum_{k=1}^{l}\left[(C_6^2 - C_9) + (2C_5 - C_8)^2/4(C_7 - C_5^2)\right]\right\} \tag{11}$$

where

$$C_1 = \frac{(2\,\mathrm{Re}\,b)^2}{8|b|^2\,\mathrm{Re}(\lambda b^*)}\bigg|_t + \frac{(2\,\mathrm{Re}\,b)^2}{8|b|^2\,\mathrm{Re}(\lambda b^*)}\bigg|_{t'}$$

$$C_2 = \frac{a_k^+ + a_k}{\sqrt{2}\,b}\bigg|_t + \frac{a_k^+ + a_k}{\sqrt{2}\,b}\bigg|_{t'}$$

$$C_3 = \frac{1}{8|b|^2\,\mathrm{Re}(\lambda b^*)}[4\,\mathrm{Re}(b^2 d^*) + 4|b|^2\,\mathrm{Re}(d)]\bigg|_t$$

$$+ \frac{1}{8|b|^2\,\mathrm{Re}(\lambda b^*)}[4\,\mathrm{Re}(b^2 d^*) + 4|b|^2\,\mathrm{Re}(d)\bigg|_{t'}$$

$$C_5 = \frac{-4\,\mathrm{Im}\,b^2}{16C_1|b|^2\,\mathrm{Re}(\lambda b^*)}\bigg|_t - \frac{4\,\mathrm{Im}\,b^2}{16C_1|b|^2\,\mathrm{Re}(\lambda b^*)}\bigg|_{t'}$$

$$C_6 = \frac{C_2 + C_3}{2C_1}$$

$$C_7 = \frac{|b|^2 - \mathrm{Re}\,b^2}{4|b|^2\,\mathrm{Re}(\lambda b^*)}\bigg|_t + \frac{|b|^2 - \mathrm{Re}\,b^2}{4|b|^2\,\mathrm{Re}(\lambda b^*)}\bigg|_{t'}$$

$$C_8 = \frac{\mathrm{Im}(b^2 d^*) - |b|^2\,\mathrm{Im}\,d}{2|b|^2\,\mathrm{Re}(\lambda b^*)}\bigg|_t$$

$$+ \frac{-\mathrm{Im}(b^2 d^*) - |b|^2\,\mathrm{Im}\,d}{2|b|^2\,\mathrm{Re}(\lambda b^*)}\bigg|_{t'}$$

$$C_9 = \frac{\mathrm{Re}(b^2 d^{*2}) + |b|^2|d|^2}{4|b|^2\,\mathrm{Re}(\lambda b^*)}\bigg|_t$$

$$+ \frac{\mathrm{Re}(b^2 d^{*2}) + |b|^2|d|^2}{4|b|^2\,\mathrm{Re}(\lambda b^*)}\bigg|_{t'}$$

In conclusion, we mention another effective approach based on the consideration of the exponential transformations enabling one to eliminate certain terms in the equations under consideration. This approach was used in the study of some models of quantum optics by Cerdeira *et al.* (1984), who analyzed the expressions for the quasienergy spectra. However, algebraic analysis of the dynamical models of quantum optics, from our point of view, presents essential advantages connected with such problems of physical importance as the construction of conservation laws, the Green function, coherent and squeezed states, possible transitions in the state of the system, and so on.

Note that generalization of the present results for the case of an electromagnetic field interaction with M-level ($M > 2$) systems requires only the modification of the structure of the magnitude g_k in (3).

ACKNOWLEDGMENTS

The author would like to thank Prof. Abdus Salam, the International Atomic Energy Agency, and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste. He would also like to thank the Condensed Matter Group for their kind support during the Research Workshop on Condensed Matter, Atomic and Molecular Physics (19 June–30 September 1989), and Profs. H. Cerdeira, Yu Lu, E. Tosatti, C. W. Lung, P. Knight, G. Barton, and F. S. Persico for a number of useful scientific discussions on the matters presented in this article.

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